1. A RELATION R FROM A TO B AGAIN FUNCTIONS JUST AS A PREDICATE, WITH $a \, R \, b$ BEING EITHER TRUE OR FALSE FOR EACH $a \in A$, $b \in B$ (EXACTLY THE SAME IDEA, BUT WITH TWO SETS).
WE CALL A THE DOMAIN AND B THE CODOMAIN OF R.

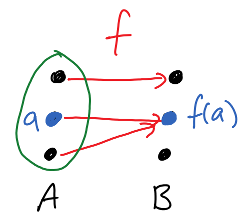
NOTE THAT BECAUSE SETS A & B COULD BE DIFFERENT, NONE OF OUR USUAL RELATION PROPERTIES (REFLEXIVE, SYMMETRIC, AND TRANSITIVE) MAKE SENSE, IN GENERAL, FOR THIS SORT OF RELATION!

2. A FUNCTION $f: A \to B$ IS A RELATION FROM A TO B WITH THE PROPERTY THAT $\forall a \in A, \exists ! \, b \in B$ WITH $a \overset{f}{\mapsto} b$.

↳ "UNIQUE" — THERE EXISTS JUST ONE SUCH $b \in B$.
FORMALLY, THIS MEANS:
① $\forall a \in A \; \exists \, b \in B$ WITH $a \overset{f}{\mapsto} b$ ("$\exists$")
AND ② $a \overset{f}{\mapsto} b \wedge a \overset{f}{\mapsto} b' \Rightarrow b = b'$ ("!")

THIS IS BEST CONCEPTUALIZED AS AN ACTIVE RULE OF ASSIGNMENT:

f "MAPS" EACH ELEMENT OF ITS DOMAIN TO EXACTLY ONE ELEMENT OF ITS CODOMAIN



IF $a \in A$, WE WRITE $f(a)$ FOR THE UNIQUE ELEMENT $b \in B$ WITH $a \overset{f}{\mapsto} b$.

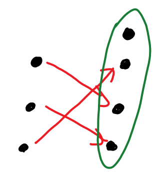
$\therefore$ "$f(x)$" ISN'T A FUNCTION — JUST AN EXPRESSION (THE VALUE f SENDS x TO).
f IS THE FUNCTION!

3. IF $f, f' : A \to B$ ARE FUNCTIONS, WE WRITE $f = f'$ WHEN $\forall a \in A, \; f(a) = f'(a)$
(IN OTHER WORDS, f & f' REPRESENT THE SAME RULE OF ASSIGNMENT)

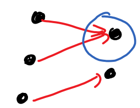
4. (a) $f: A \to B$ IS INJECTIVE MEANS $f(a)=f(a') \Rightarrow a=a'$ IN TERMS OF FORMAL LOGIC.

INTUITIVELY, IT MEANS THAT EVERY ELEMENT $b \in B$ GETS LANDED ON BY AT MOST ONE ELEMENT OF A, OR THAT AN "f" AS THE LAST OPERATION ON BOTH SIDES OF AN EQUATION CAN BE PEELED OFF.

THIS IS EFFECTIVELY THE CONVERSE OF WHAT IT MEANS TO BE A FUNCTION, I.E., $a=a' \Rightarrow f(a)=f(a')$: A FUNCTION f CAN BE APPLIED TO BOTH SIDES OF AN EQUATION!



INJECTIVE

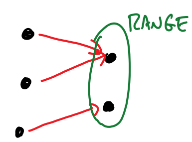


NOT INJECTIVE

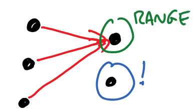
(b) $f: A \to B$ IS SURJECTIVE MEANS $\forall\, b \in B\ \exists\, a \in A$ WITH $f(a)=b$ IN TERMS OF FORMAL LOGIC.

INTUITIVELY, THIS MEANS THAT EVERY ELEMENT $b \in B$ GETS LANDED ON BY AT LEAST ONE ELEMENT OF A.

THE RANGE OF $f = \{f(a) : a \in A\}$ OR, MORE FORMALLY, $\{b \in B : \exists\, a \in A$ WITH $f(a)=b\}$; THIS GIVES THE SUBSET OF B CONSISTING OF ALL ELEMENTS LANDED ON BY SOME $a \in A$.



SURJECTIVE

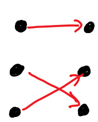


NOT SURJECTIVE

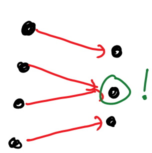
$\therefore f$ IS SURJECTIVE $\Leftrightarrow$ THE RANGE OF f IS ALL OF B.

(c) $f: A \to B$ IS BIJECTIVE MEANS THAT f IS BOTH INJECTIVE & SURJECTIVE.

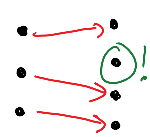
INTUITIVELY, EACH $b \in B$ GETS LANDED ON BY EXACTLY ONE ELEMENT $a \in A$, AND EACH $a \in A$ GETS SENT TO EXACTLY ONE ELEMENT OF B, SO A BIJECTIVE FUNCTION (A BIJECTION) "PAIRS UP" EACH ELEMENT OF A WITH ONE OF B, AND VICE-VERSA.



BIJECTIVE
(BOTH INJECTIVE & SURJECTIVE)



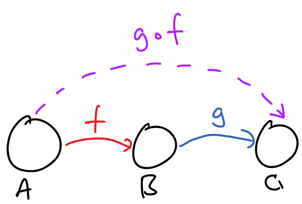
NOT BIJECTIVE
(NOT INJECTIVE)



ALSO NOT BIJECTIVE
(NOT SURJECTIVE)

5. IF $f: A \to B$ AND $g: B \to C$ ARE FUNCTIONS, WE DEFINE THEIR COMPOSITION $g \circ f: A \to C$ BY $a \xmapsto{g \circ f} g(f(a))$, I.E., $(g \circ f)(a) = g(f(a))$. (APPLY f, THEN g)

$\left(\substack{g \circ f \ldots \\ \text{OF } a} \right)$ IS $\left(\substack{g \text{ OF} \ldots \\ f(a)} \right)$



THE DOMAIN OF $g \circ f$ IS THE DOMAIN OF f, AND THE CODOMAIN OF $g \circ f$ IS THE CODOMAIN OF g.

6. IF X IS ANY SET, THE IDENTITY FUNCTION $id_X : X \to X$ IS GIVEN BY $x \xmapsto{id_X} x$
(I.E., MAP EACH $x \in X$ TO ITSELF!)

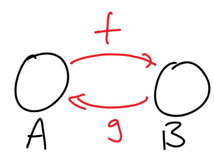
IT IS EASILY SHOWN VIA THE DEFINITIONS THAT id_X IS BIJECTIVE.

IT IS VERY QUICK TO CHECK THAT IDENTITY FUNCTIONS EVAPORATE FROM COMPOSITIONS (BECAUSE THEY DON'T DO ANYTHING!):

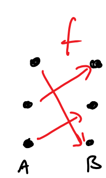
$$\text{IF } f: A \to B, \text{ THEN } \quad f \circ \cancel{id_A} = f \quad \text{AND} \quad \cancel{id_B} \circ f = f$$

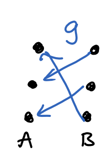
7. FOR $f: A \to B$ AND $g: B \to A$ TO BE INVERSES FORMALLY MEANS

$$\left.\begin{array}{ll} \bullet\ \forall a \in A, & g(f(a)) = a \\ \text{AND } \bullet\ \forall b \in B, & f(g(b)) = b. \end{array}\right\} (*)$$



INTUITIVELY, THE FUNCTIONS f & g UNDO EACH OTHER: g SENDS EACH $b \in B$ BACK WHERE IT CAME FROM VIA f, AND f SENDS EACH $a \in A$ BACK WHENCE IT CAME VIA g.





SYMBOLICALLY, INVERSE FUNCTIONS CAN BE CANCELED WHEN THEY ARE APPLIED IN IMMEDIATE SUCCESSION, IN EITHER ORDER!

$$\cancel{g}(\cancel{f}(a)) = a$$
$$\cancel{f}(\cancel{g}(b)) = b$$

THE TWO HALVES OF THE FORMAL DEFINITION (*) CAN BE PHRASED VERY SUCCINCTLY IN TERMS OF COMPOSITIONS: $g \circ f = id_A$ AND $f \circ g = id_B$

8. SUPPOSE THAT $f: A \to B$ AND $g: B \to C$ ARE FUNCTIONS.

$\star\ (g\circ f)(a)=(g\circ f)(a') \Rightarrow a=a'$

(a) CLAIM: f, g INJECTIVE $\Rightarrow g\circ f$ INJECTIVE.

PROOF: SUPPOSE f IS INJECTIVE, I.E., $f(a)=f(a') \Rightarrow a=a'$ $(*)$

& g IS INJECTIVE, I.E., $g(b)=g(b') \Rightarrow b=b'$ $(**)$

USE APPROPRIATE VARIABLES FOR THE SETS' ELEMENTS, THOUGH IT TECHNICALLY DOESN'T MATTER!

SUPPOSE $(g\circ f)(a)=(g\circ f)(a')$,

I.E. $g(f(a))=g(f(a'))$

SO $f(a)=f(a')$ BY $(**)$ (PEEL AWAY THE g'S)

AND THUS $a=a'$ BY $(*)$ ∎

$\star\ f(a)=f(a') \Rightarrow a=a'$

(b) CLAIM: $g\circ f$ INJECTIVE $\Rightarrow f$ INJECTIVE.

PROOF: SUPPOSE $g\circ f$ IS INJECTIVE, I.E., $(g\circ f)(a)=(g\circ f)(a') \Rightarrow a=a'$ $(*)$

IN C

SUPPOSE $f(a)=f(a')$. ← IN B

APPLY g TO BOTH SIDES: $g(f(a))=g(f(a'))$

I.E., $(g\circ f)(a)=(g\circ f)(a')$

SO $a=a'$ BY $(*)$ ∎

$\star\ \forall c\in C\ \exists a\in A$ WITH $(g\circ f)(a)=c$

(c) CLAIM: f, g SURJECTIVE $\Rightarrow g\circ f$ SURJECTIVE.

PROOF: SUPPOSE f IS SURJECTIVE, I.E., $\forall b\in B\ \exists a\in A$ WITH $f(a)=b$ $(*)$

AND g IS SURJECTIVE, I.E., $\forall c\in C\ \exists b\in B$ WITH $g(b)=c$ $(**)$

SHOULD WE REPEAT THIS VARIABLE? HERE, YES, BECAUSE WE'LL USE EXACTLY THE SAME "b" IN BOTH

LET $c\in C$ BE GIVEN

BY $(**)$, $\exists\ b\in B$ WITH $g(b)=c$. $(\dagger)$

TAKE THIS $b\in B$; THEN BY $(**)$, $\exists\ a\in A$ WITH $f(a)=b$. $(\dagger\dagger)$

TAKE THIS $a\in A$. THEN $(g\circ f)(a) = g(f(a))$

$= g(b)$ BY $(\dagger\dagger)$

$= c.$ BY $(\dagger)$ ∎

(d) CLAIM: $g \circ f$ SURJECTIVE $\Rightarrow$ g SURJECTIVE $\Leftrightarrow \forall\, c \in C\ \exists\, b \in B$ WITH $g(b)=c$

PROOF: SUPPOSE $g \circ f$ IS SURJECTIVE, I.E., $\underbrace{\forall\, c \in C\ \exists\, a \in A \text{ WITH } (g\circ f)(a)=c}_{(*)}$

LET $c \in C$ BE GIVEN.

THEN BY $(*)$, $\exists\, a \in A$ WITH $(g \circ f)(a) = c$

TAKE THIS $a \in A$. THEN $(g\circ f)(a)=c$, SO $g(f(a)) = c$.

TAKE $b = f(a) \in B$. THEN $g(b) = g(f(a)) = c$. ■

9. IF $f: A \to B$ AND $g: B \to A$ ARE INVERSES, THEN:

- $g \circ f = id_A$, WHICH IS BOTH INJECTIVE & SURJECTIVE
  - $\Rightarrow f$ INJECTIVE, BY 8(b)
  - $\Rightarrow g$ SURJECTIVE, BY 8(d)

AND
- $f \circ g = id_B$, WHICH IS BOTH INJECTIVE & SURJECTIVE.
  - $\Rightarrow g$ INJECTIVE, BY 8(b)
  - $\Rightarrow f$ SURJECTIVE, BY 8(d)

IN SUMMARY, IF f AND g ARE INVERSES, THEN f AND g ARE BOTH BIJECTIVE.

10. SUPPOSE THAT g, g' ARE BOTH INVERSES TO $f: A \to B$.

$$\left.\begin{array}{l} \text{THEN } g \circ f \circ g' = g \circ (\cancel{f \circ g'}) = g \circ \cancel{id_B} = g \\ \text{AND } g \circ f \circ g' = (\cancel{g \circ f}) \circ g' = \cancel{id_A} \circ g' = g' \end{array}\right\} \therefore\ g = g'$$

WE THEN DENOTE THE INVERSE OF f BY f^{-1}.

THE SUPERSCRIPT -1 HERE DENOTES THE INVERSE FUNCTION, NOT RECIPROCATION! THE INVERSE OF f EXISTS IF, AND ONLY IF, f IS BIJECTIVE!

11. IF $f: A \to B$ AND $g: B \to C$ ARE BOTH INVERTIBLE, THEN THEY ARE BOTH BIJECTIVE, SO BY 8(a,c), THEIR COMPOSITION $g \circ f$ IS ALSO BIJECTIVE, AND THUS INVERTIBLE. THE FORMULA FOR ITS INVERSE IS $(g\circ f)^{-1} = f^{-1} \circ g^{-1}$, BECAUSE:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[rr, bend left=60, dashed, "g\circ f"] & B \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g"] & C \arrow[l, bend left, "g^{-1}"] \arrow[ll, bend left=60, dashed, "(g\circ f)^{-1} = f^{-1}\circ g^{-1}"] \end{tikzcd}$$

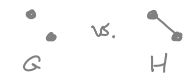
- $(g\circ f)\circ(f^{-1}\circ g^{-1}) = g \circ (\cancel{f \circ f^{-1}}) \circ g^{-1} = g \circ g^{-1} = id_C$

AND
- $(f^{-1}\circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (\cancel{g^{-1} \circ g}) \circ f = f^{-1} \circ f = id_A$.

12. GRAPH $G \overset{v}{\longmapsto}$ # OF VERTICES IN G

(a) IF G AND H ARE ISOMORPHIC, THEN THE VERTICES OF G CAN BE PAIRED UP WITH THOSE OF H, SO $\underbrace{v(G) = v(H)}$.

(b) CONSEQUENTLY, IF $v(G) \neq v(H)$, THEN G & H CANNOT BE ISOMORPHIC. (CONTRAPOSITIVE OF (a)!)

(c) IF $v(G) = v(H)$, NO CONCLUSION CAN BE DRAWN. (E.G., 

)

(d) E.G.: # OF EDGES; # OF COMPONENTS; SET OF ALL DEGREES OF VERTICES (OR BETTER YET, THE LIST OF ALL VERTEX DEGREES, IN INCREASING ORDER!)

$\boxed{\text{BUT NO KNOWN COMBINATION OF SUCH SIMPLE INVARIANTS WILL GUARANTEE THAT TWO GRAPHS } \underline{\text{ARE}} \text{ ISOMORPHIC — THIS WOULD BE A } \underline{\text{PERFECT INVARIANT}} \text{ AND WOULD SOLVE THE ISOMORPHISM PROBLEM FOR GRAPHS!}}$